MATH3060 HW6 Due Date : Nov 9, 2016

1. Show that an orthonormal set $\{\varphi_k\}_{k=1}^{\infty}$ in $C[a,b]$ under L^2-product has no convergent subsequence in the L^2-metric.

2. Using open cover description, show that every continuous mapping in a compact set in some metric space (X,d) to another metric space (Y,ρ) is uniformly continuous.

3. Show that every continuous function from a compact metric space (X,d) to $\mathbb{R}$ attains its minimum and maximum.

4. Let (X,d) be a metric space and $C_b(X)$ the vector space of all bounded continuous functions in X. Show that $(C_b(X), d_\infty)$ is a complete metric space, where

$$d_\infty(f,g) = \sup_X |f-g| .$$

5. Let (X,d) be a metric space and $p \in X$ is a fixed point. Define for each $x \in X$, the function $f_x : X \to \mathbb{R}$ by

$$f_x(y) = d(y,x) - d(y,p).$$

(a) Show that $f_x \in C_b(X)$, where $C_b(X)$ as in question (4).

(b) Show that the mapping $\Phi : (X,d) \to (C_b(X), d_\infty)$ defined by $\Phi(x) = f_x \in C_b(X)$ is an isometric embedding.

6. Let T be a continuous self map of a complete metric space (X, d). Suppose that for some $k \geq 1$, T^k is a contraction. Show that T admits a unique fixed point.

7. Consider maps from $\mathbb{R}$ to itself. Find an explicit example of a map satisfying $|f(x) - f(y)| < |x - y|$ but with no fixed point.